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Cayley Parametrization of Semisimple Lie Groups and its Application to Physical Laws in a (3 + 1)-Dimensional Cubic Lattice†

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Abstract

The assumption of a discrete space-time is expressed mathematically by restricting the space-time variables to the field of integer numbers, and by restricting to the field of rational numbers the functions describing the laws of motion. This rational character must be preserved under the transformations connecting different systems of reference. The Cayley parametrization of semisimple Lie groups, and in particular of the Lorentz group, satisfies this condition if we require these parameters to take only integer values. The rational points of the most frequently used transcendental functions are obtained with the help of the integer complex and hypercomplex numbers. Some applications are made concerning the laws of motion in special relativity defined over a (3 + 1)-dimensional cubic lattice.

1. *Introduction*

The idea of a discrete space-time has been introduced by physicists in the past in several different ways.

Heisenberg (1938, 1943) advocated a fundamental length and inferred its connection with a discrete mass spectrum.

Snyder (1947) has proposed a quantized space-time in the sense of coordinate operators with discrete spectrum, but the introduction of a finite minimal unit of length forces the non-commutativity of these operators.

Flint & Williamson (1953) modified Snyder's position operator by using an elementary length in the direction of motion.

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According to Darling (1950) the space-time is continuous but the dynamical laws must be expressed by means of finite difference equations, a method which was also adopted by Hellend & Tanaka (1954).

In a different approach, Castell (1966, 1967) assumed that the space-time structure at the microscopic level is determined by a fundamental length. Led by symmetry considerations, he postulated that this microscopic length is the scale constant of a 5-dimensional pseudohyperbolic geometry embedded in a 6-dimensional Euclidean space.

In the construction of unitary irreducible representations of a new dynamical group, Aghassi, Roman & Santilli (1970, 1971) found a fundamental length associated with the central extension of the covering group (which length gives at the same time one of the labels characterizing the irreducible representations) and proposed a covariant four position operator which belongs to the associated Lie algebra.

In the aforementioned papers the quantization of space-time variables has been obtained, in general terms, by the introduction of difference operators or quantum generators. Recently Ahmavaara (1965, 1966) proposed a finite space-time cubic lattice, which is embedded in a finite linear space over a Galois field. Similarly, Bopp (1967) adopted the idea of a finite cubic lattice, with discrete space variables and continuous time, but this lattice structure is Lorentz invariant only when the number of points becomes infinite. Finally, Greenspan (1973) expressed the law of classical mechanics in difference equations with discrete space-time variables, and these equations are invariant under continuous groups of transformations.

In this paper the assumption of a lattice structure for the points of space-time is adopted, which requires the use of difference equations. A stronger assumption requires the solutions of these difference equations to take only rational values. As a consequence of the conservation laws, this lattice structure and the rational character should be preserved under coordinate transformations without taking the limit to a continuous structure of the space-time.

The strong character of these assumptions makes it very hazardous to accept them, because of the possibility of non-physical constraints; but at the same time it opens the way to new superselection rules to fit some discrete values of physical magnitudes.

In this paper simple consequences and elementary methods that arise from these assumptions have been elaborated in a more intuitive than rigorous way, and a few simple examples in the area of classical relativistic mechanics and electrodynamics are given. Yet no applications to the problems of quantization will be made, which nevertheless seems to be the crucial test of the assumptions.

In Sections 2 to 5 a classical treatment of the Cayley parametrization of the semisimple Lie groups is made with emphasis on the proper rotation and Lorentz groups. Using the multiplication law of the Cayley parameters of the Lorentz group, an associative non-division algebra of hypercomplex numbers is introduced.

In Section 6, the assumptions adopted and the invariance principles connected with them are explained.

In Sections 7 and 8, a method is constructed for finding the rational points of the trigonometric and hyperbolic functions and other quadatic forms with the help of the hypercomplex numbers introduced earlier.

In Section 9, the Cayley parametrization is applied to the calculation of the rational matrix elements of the semisimple groups.

In Section 10, simple examples of equations of motion in relativistic mechanics and electrodynamics are worked out with the help of the mathematical tools introduced in previous paragraphs.

2. *Cayley's Rational Parametrization of Semisimple Groups*

Let $\mathscr{A}$ be a semisimple Lie group of complex matrices A, which leaves invariant some non-degenerate bilinear form.

We call a matrix A of the group $\mathscr{A}$ non-exceptional if $\det(E+A) \neq 0$, where E is the unit matrix. Cayley (1846) has proved that every non-exceptional matrix A can be expressed as follows

$$A = (E+S)^{-1}(E-S) = (E-S)(E+S)^{-1} \tag{2.1}$$

where S is also a non-exceptional matrix.

If G is the coefficient matrix of the non-degenerate bilinear form, which is left invariant under the group $\mathscr{A}$, the non-exceptional matrices A satisfy the relation

$$A^{+}GA = G \tag{2.2}$$

and because of (2.1) the corresponding matrices S will also satisfy

$$S^{+}G + GS = 0 \tag{2.3}$$

In order to obtain the independent parameters of the semisimple group $\mathscr{A}$ it is more convenient to work with expression (2.3), which is linear, rather than with expression (2.2), which is quadratic. If we diagonalize or reduce to the canonical form the coefficient matrix G we have a further simplification of (2.3). Taking the independent elements of the matrix S given by (2.3) to be the independent parameters, we obtain Cayley's rational parametrization of the semisimple group $\mathscr{A}$. (Note that when the independent elements of S are complex their real and imaginary part should be taken as independent parameters.)

In Table 1 we give the explicit conditions on the non-exceptional matrices A and S for all semisimple Lie groups, as derived from expression (2.2) and (2.3), respectively. The notation A^T means the transpose matrix and A^{+} the adjoint. Also

$$J \equiv \left(\begin{array}{c|c} 0 & E_n \\ \hline -E_n & 0 \end{array}\right)$$

in the group $Sp(2n)$ and

$$I \equiv \left(\begin{array}{c|c} E_p & 0 \\ \hline 0 & -E_q \end{array}\right)$$

in the groups $SO(p, q)$ and $SU(p, q)$, where E_n, E_p, E_q are the unit matrix of order n, p, q respectively. The condition on the matrix S gives automatically the unimodularity condition

$$\det(E + A) = \det(E - A) \tag{2.4}$$

except in the groups $SU(n + 1)$ and $SU(p, q)$ and therefore (2.4) imposes an extra condition on the parameters corresponding to these groups.

TABLE 1. Cayley's decomposition of semisimple groups

Group	Conditions on A	Conditions on S	Unimodularity	Parameters
$SO(2n)$	$A^TA = E$	$S^T + S = 0$		$n(2n - 1)$
$SO(2n + 1)$	$A^TA = E$	$S^T + S = 0$		$n(2n + 1)$
$SU(n + 1)$	$A^+A = E$	$S^+ + S = 0$	$\|E + A\| = \|E - A\|$	$n(n + 2)$
$Sp(2n)$	$A^TJA = J$	$S^TJ + JS = 0$		$n(2n + 1)$
$SO(p, q)$	$A^TIA = I$	$S^TI + IS = 0$		$1/2(p+q)(p+q-1)$
$SU(p, q)$	$A^+IA = I$	$S^+I + IS = 0$	$\|E + A\| = \|E - A\|$	$(p + q)^2 - 1$

When the matrix A is unimodular but exceptional, i.e. when $\det(E + A) = 0$, then Cayley's decomposition (2.1) is not possible, but in this case Weyl (1946) has proved that any exceptional unimodular matrix A can be transformed into the form

$$A = \left(\begin{array}{c|c} -E_{2p} & 0 \\ \hline 0 & B \end{array}\right)$$

where E_{2p} is a unit matrix of even dimension and B is a non-exceptional matrix. Moreover the matrix A can be expressed as the product of two commuting non-exceptional matrices.

Some useful expressions derived from (2.1) are

$$A = \frac{E - S}{E + S} = \frac{2E}{E + S} - E \tag{2.5}$$

where the symbol of division has been used, because the matrices of the numerator and of the denominator commute. For any non-singular matrix B we have

$$BAB^{-1} = \frac{E - BSB^{-1}}{E + BSB^{-1}}$$

Also from the product of two non-exceptional matrices $A_1A_2 = A$ one obtains

$$E + S = (E + S_2)(E + S_1S_2)^{-1}(E + S_1) \tag{2.6}$$

which yields the multiplication law for the parameters of the group, i.e., the Cayley parameters of the matrix A in terms of the parameters of the matrix A_1 and A_2. In particular, if

$$[A_1, A_2] = 0, \qquad \text{then } S = \frac{S_1 + S_2}{E + S_1 S_2}$$

if

$$[S_1, S_2] = S_1^2 = S_2^2 = 0, \qquad \text{then } S = S_1 + S_2$$

3. *Some Examples*

3.1. *The Rotation Group, SO(3)*

From Table 1 the matrix S is antisymmetric and it can be expressed in the following way

$$S = \frac{1}{m}\begin{pmatrix} 0 & n & -p \\ -n & 0 & q \\ p & -q & 0 \end{pmatrix} \tag{3.1.1}$$

where n, p, q are independent parameters and m has been introduced for convenience. Using (2.5) and (3.1.1) one obtains the Cayley parametrization of the non-exceptional matrix of the rotation group

$$A = \frac{1}{m^2 + n^2 + p^2 + q^2} \times \begin{pmatrix} m^2 - n^2 - p^2 + q^2 & -2mn + 2pq & 2mp + 2nq \\ 2mn + 2pq & m^2 - n^2 + p^2 - q^2 & -2mq + 2np \\ -2mp + 2nq & 2mq + 2np & m^2 + n^2 - p^2 - q^2 \end{pmatrix} \tag{3.1.2}$$

If we define $\alpha = m + in, \beta = p - iq$ and then impose $m^2 + n^2 + p^2 + q^2 = 1$, the parametrization of the matrix A given by (3.1.2) is identical with the parametrization used by Wigner (1959a) for the 3-dimensional rotation group. The parameters α and β used by him are related to the parametrization of $SU(2)$, the covering group of $SO(3)$, in this way

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{3.1.3}$$

The one-to-two correspondence between $SO(3)$ and $SU(2)$ groups is obvious: two different elements A and $-A$ of the group $SU(2)$ defined by (m, n, p, q)

and $(-m, -n, -p, -q)$ correspond to only one element of the rotation group.

The exceptional proper matrices of the rotation group are

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

and all the matrices obtained from these by a similarity transformation with any of the non-exceptional proper matrices (3.1.2).

In terms of the components of the axis of rotation (a_1, a_2, a_3) and of the angle of rotation ϕ, the Cayley parameters have the following geometrical interpretation

$$\frac{q}{a_1} = \frac{p}{a_2} = \frac{n}{a_3}, \qquad \cos\phi = \frac{m^2 - n^2 - p^2 - q^2}{m^2 + n^2 + p^2 + q^2} \tag{3.1.4}$$

When $\phi = \pi$, then from (3.1.4) we obtain either $m = 0$, or some of the parameters n, p, q go to infinity. Since the exceptional matrices correspond also to $\phi = \pi$, it follows that in the case of non-exceptional matrice (3.1.2) the parameters n, p, q must be finite and $m \neq 0$.

Similar parametrization and corresponding properties can be obtained for the N-dimensional rotation groups.

3.2. *The Unitary Group SU*(2)

From Table 1 the matrix S is antihermitian and it can be expressed as

$$S = \frac{1}{l}\begin{pmatrix} ia & \rho \\ -\rho^* & ib \end{pmatrix} \tag{3.2.1}$$

where a and b are real parameters, $p = r + is$ and l has been added for convenience.

From (2.5) and (3.2.1) one obtains

$$A = \frac{1}{\Delta}\begin{pmatrix} l^2 + ab - |\rho|^2 + i2lb & -2l\rho \\ 2l\rho^* & l^2 + ab - |\rho|^2 + i2la \end{pmatrix} \tag{3.2.2}$$

with $\Delta = l^2 - ab + |\rho|^2 + il^2(a + b)$.

The antihermiticity of S gives

$$\det(E + S)^* = \det(E - S)$$

but it does not imply the unimodularity condition. (In the rotation group the antisymmetry of S does imply the unimodularity of A.) If we impose $\det A = 1$, from (2.1) follows that

$$\det(E+S) = \det(E-S)$$

Both conditions, unitarity and unimodularity of A, give $\det(E+S) =$ real, or

$$a + b = \operatorname{Tr} S = 0 \tag{3.2.3}$$

Substituting (3.2.3) in (3.2.2) we obtain the general expression for the unitary unimodular matrices in two dimensions

$$A = \frac{1}{\Delta}\begin{pmatrix} l^2 - a^2 - r^2 - s^2 - i2la & -2lr - i2ls \\ 2lr - i2ls & l^2 - a^2 - r^2 - s^2 + i2la \end{pmatrix} \tag{3.2.4}$$

with $\Delta = l^2 + a^2 + r^2 + s^2$. Obviously the matrix (3.2.4) is equivalent to (3.1.3), but uses different parametrization.

3.3. *The Unitary Group $SU(3)$*

From Table 1 the matrix S is the general 3-dimensional hermitian matrix

$$S = \frac{1}{l}\begin{pmatrix} ia & \rho & \sigma \\ -\rho^* & ib & \tau \\ -\sigma^* & -\tau^* & ic \end{pmatrix} \tag{3.3.1}$$

where a, b, c are real parameters, ρ, σ, τ are complex and l is introduced for the sake of homogeneity. As in the case of the $SU(2)$ group, and in contrast with the orthogonal group, the unimodularity condition is not implied by (3.3.1). If we impose the last condition together with the unitarity of A, we have

$$\det(E+S) = \text{real} \qquad \text{or} \qquad i(a+b+c) + l\det S = 0 \tag{3.3.2}$$

which restricts to eight the number of independent parameters. Observe that in this case the matrix S is not traceless, contrary to the case of the orthogonal and $SU(2)$ groups. We will come back later to this unwanted result, because it is desirable to have the matrices S with the same properties of the corresponding infinitesimal generators.

3.4. *The Proper Lorentz Group $SO(3.1)$*

From Table 1 one obtains the traceless matrix

$$S = \frac{1}{m}\begin{pmatrix} 0 & n & -p & r \\ -n & 0 & q & s \\ p & -q & 0 & t \\ r & s & t & 0 \end{pmatrix} \tag{3.4.1}$$

where n, p, q, r, s, t are real independent parameters and m has been introduced as before. The unimodularity of A does not impose further conditions on these parameters. From (2.5) one gets

$$A = \frac{1}{\Delta}\begin{bmatrix} m^2 - n^2 - p^2 + q^2 + r^2 - s^2 - t^2 + \lambda^2 & -2mn + 2pq + 2rs + 2\lambda t & 2mp + 2nq + 2rt - 2\lambda s & -2mr + 2ns - 2pt - 2\lambda q \\ 2mn + 2pq + 2rs - 2\lambda t & m^2 - n^2 + p^2 - q^2 - r^2 + s^2 - t^2 + \lambda^2 & -2mq + 2np + 2st + 2\lambda r & -2ms - 2nr + 2qt - 2\lambda p \\ -2mp + 2nq + 2rt + 2\lambda s & 2mq + 2np + 2st - 2\lambda r & m^2 + n^2 - p^2 - r^2 - s^2 + t^2 + \lambda^2 & -2mt + 2pr - 2qs - 2\lambda n \\ -2mr - 2ns + 2pt - 2\lambda q & -2ms + 2nr - 2qt - 2\lambda p & -2mt - 2pr + 2qs - 2\lambda n & m^2 + n^2 + p^2 + q^2 + r^2 + s^2 + t^2 + \lambda^2 \end{bmatrix} \tag{3.4.2}$$

where $m\lambda = nt + ps + qr$

$$\Delta = m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2$$

If $\Delta > 0$, since $\det A = 1$, one obtains the general expression for the non-exceptional matrices of the proper Lorentz group ($A_{44} > 0$).

If $r = s = t = 0$, one recovers expression (3.1.2) for the proper orthogonal group in 3-dimensions.

If $n = p = q = 0$ one is left with the non-exceptional matrices of the pure Lorentz transformations. In this case, comparison of (3.4.2) with a pure Lorentz transformation with velocity v along $\mathbf{v}$ gives (Møller, 1952)

$$\frac{v_x}{r} = \frac{v_y}{s} = \frac{v_z}{t} = \frac{2mc}{m^2 + r^2 + s^2 + t^2}, \qquad \left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{m^2 - r^2 - s^2 - t^2}{m^2 + r^2 + s^2 + t^2} \tag{3.4.3}$$

where c is the velocity of light in vacuum.

If $m = 0$ or r, s, t goes to infinity, we have from (3.4.2) the following matrices

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

which are exceptional and also do not belong to the proper Lorentz group ($A_{44} < 0$). Therefore for the non-exceptional matrices of the proper Lorentz group the parameters r, s, t must be finite (n, p, q must also be finite as we have seen before) and $m \neq 0$.

We still can have exceptional matrices of the proper Lorentz group such as

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & -1 & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

with $a_{ii} = a_{44}$ and $a_{i4} = a_{4i}, a_{ii}^2 - a_{i4}^2 = 1$ ($i = 1, 2, 3$) and all the matrices obtained by these by a similarity transformation with the help of (3.4.2).

If we define

$$\left.\begin{aligned} \alpha &= m - t + i(n - \lambda), & \beta &= -p - r + i(q - s) \\ \gamma &= p - r + i(q + s), & \delta &= m + t - i(n + \lambda) \end{aligned}\right\} \qquad (3.4.4)$$

and introduce these variables in the general expression of the proper Lorentz group in terms of the parameters of the $SL(2, C)$ group (Naimark, 1964a)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1$$

we obtain the expression (3.4.2) plus the condition

$$\left.\begin{aligned} m\lambda &= nt + ps + qr \\ \Delta &= m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2 = 1 \end{aligned}\right\} \qquad (3.4.5)$$

As in the case of the rotation group, the one to two correspondence between the proper Lorentz group and its covering group, $SL(2,C)$ can be easily seen with this parametrization; two different elements, A and $-A$ of the group $SL(2, C)$ defined by the sets (m, n, p, q, r, s, t) and $(-m, -n, -p, -q, -r, -s, -t)$ correspond to one and the same matrix of the proper Lorentz group.

4. *Hypercomplex Numbers Associated With Cayley Parameters of the Lorentz Group*

The general composition law of the Cayley parameters can be found from (2.6). However, in the case of the proper Lorentz group, it takes a particular simple form due to the homomorphism between this group and $SL(2,C)$. In fact, substituting (3.4.4) in the multiplication law of $SL(2,C)$, namely,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}=\begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix} \qquad \alpha\delta-\beta\gamma=1$$

and comparing the matrix elements on both sides, we easily obtain

$$\begin{bmatrix} m'' \\ n'' \\ p'' \\ q'' \\ r'' \\ s'' \\ t'' \\ \lambda'' \end{bmatrix}=\begin{bmatrix} m & -n & -p & -q & r & s & t & -\lambda \\ n & m & q & -p & s & -r & \lambda & t \\ p & -q & m & n & -t & \lambda & r & s \\ q & p & -n & m & \lambda & t & -s & r \\ r & s & -t & -\lambda & m & -n & p & -q \\ s & -r & -\lambda & t & n & m & -q & -p \\ t & -\lambda & r & -s & -p & q & m & -n \\ \lambda & t & s & r & q & p & n & m \end{bmatrix}\begin{bmatrix} m \\ n \\ p' \\ q' \\ r' \\ s' \\ t' \\ \lambda' \end{bmatrix} \tag{4.1}$$

with the parameters satisfying (3.4.5). The square matrix U of expression (4.1) is itself a non-unitary 8-dimensional representation of the proper Lorentz group. In order to ascertain whether this representation is irreducible we calculate the infinitesimal generators $J_{\mu\nu}$ that satisfy the standard commutation relations, and substitute them into the Casimir operator. We find

$$\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}=l_0^2+l_1^2-1=\tfrac{6}{4} \tag{4.2}$$

where (l_0, l_1) are the characteristic labels of some irreducible representation. Here, the only possible solutions for (l_0, l_1) are $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, -\frac{3}{2})$, which correspond to the 2-dimensional irreducible spinor representations. Hence, our matrix U is a reducible representation of the proper Lorentz group.

With the help of the matrix U we can also construct a system of hypercomplex numbers in the following way: Let us expand the matrix U as a linear combination of 8 numerical matrices, each multiplied by one of the 8 parameters (we also relax the conditions (3.4.5) on these parameters):

$$U=mu_0+nu_1+pu_2+qu_3+ru_4+su_5+tu_6+\lambda u_7 \tag{4.3}$$

The matrix multiplication of any two matrices u_A $(A=0,1,\ldots 7)$ is given in Table 2. (As usual, one matrix in the left side multiplied by one matrix in the upper side gives the matrix in the intersection.)

Choosing the u_A as basis elements and the multiplication law their matrix multiplication, we can construct an associative algebra over the field of real numbers; in other words, we have a field of hypercomplex numbers, defined by (4.3),

TABLE 2. Multiplication table of the basis elements u_A

	1	u_1	u_2	u_3	u_4	u_5	u_6	u_7
1	1	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	u_1	-1	$-u_3$	u_2	u_5	$-u_4$	u_7	$-u_6$
u_2	u_2	u_3	-1	$-u_1$	$-u_6$	u_7	u_4	$-u_5$
u_3	u_3	$-u_2$	u_1	-1	u_7	u_6	$-u_5$	$-u_4$
u_4	u_4	$-u_5$	u_6	u_7	1	$-u_1$	u_2	u_3
u_5	u_5	u_4	u_7	$-u_6$	u_1	1	$-u_3$	u_2
u_6	u_6	u_7	$-u_4$	u_5	$-u_2$	u_3	1_3	u_1
u_7	u_7	$-u_6$	$-u_5$	$-u_4$	u_3	u_2	u_1	-1

with real components $(m, n, p, q, r, s, t, \lambda)$ and generators u_A $(A = 0, \dots 7)$ satisfying the multiplication law given by Table 2. This algebra is not a division algebra, because it has divisors of zero. A 2-dimensional representation of this algebra can be obtained with the help of (3.4.4)

$$\left.\begin{aligned}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} &= m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + n\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} + p\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + q\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\\ &+ r\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} + s\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + t\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + \lambda\begin{pmatrix}-i & \\ & -i\end{pmatrix}\end{aligned}\right\} \tag{4.4}$$

Identifying (4.4) and (4.3) it can be checked that these 2-dimensional matrices satisfy the multiplication law of Table 2.

5. *Cayley Parameters and Generalized Euler Angles*

According to the geometrical interpretation of the Cayley parameters of the rotation group given by (3.1.4), if we take $p = q = 0$ in (3.1.2), the matrix

$$A = \frac{1}{m^2 + n^2}\begin{pmatrix} m^2 - n^2 & -2mn & 0\\ 2mn & m^2 - n^2 & 0\\ 0 & 0 & m^2 + n^2\end{pmatrix} \tag{5.1}$$

represents a rotation around the x_3-axis. If we compare this matrix with the matrix that gives the same rotation in terms of the angle of rotation ϕ we obtain the relation between the two kinds of parameters

$$\cos\phi = \frac{m^2 - n^2}{m^2 + n^2} \qquad \sin\phi = \frac{2mn}{m^2 + n^2} \tag{5.2}$$

with $-\pi < \phi < \pi$ and $m \neq 0$, $-\infty < m, n < \infty$.

In the same way the Euler decomposition of the general element of the rotation group can be achieved in terms of the Cayley parameters

$$A = \frac{1}{\Delta}\begin{pmatrix} m_1^2 - n_1^2 & -2m_1n_1 & 0 \\ 2m_1n_1 & m_1^2 - n_1^2 & 0 \\ 0 & 0 & m_1^2 + n_1^2 \end{pmatrix}\begin{pmatrix} m_2^2 + n_2^2 & 0 & 0 \\ 0 & m_2^2 - n_2^2 & -2m_2n_2 \\ 0 & 2m_2n_2 & m_2^2 + n_2^2 \end{pmatrix}$$

$$\times \begin{pmatrix} m_3^2 - n_3^2 & -2m_3n_3 & 0 \\ 2m_3n_3 & m_3^2 - n_3^2 & 0 \\ 0 & 0 & m_3^3 + n_3^2 \end{pmatrix} \qquad (5.3)$$

where $\Delta = (m_1^2 + n_1^2)(m_2^2 + n_2^2)(m_3^2 + n_3^2)$.

Symbolically, if we represent the general matrix of the rotation group (3.1.2) by $A(m, n, p, q)$, the Euler decomposition can be written

$$A(m, n, p, q) = A(m_1, n_1, 0, 0)A(m_2, 0, 0, n_2)A(m_3, n_3, 0, 0) \qquad (5.4)$$

The relation between the Cayley parameters in the Euler decomposition (5.3) and the parameters of the general rotation (3.1.2) is easily found with the help of the multiplication law (4.1), namely,

$$\left.\begin{aligned} m &= m_1m_2m_3 - n_1m_2n_3, & n &= n_1m_2m_3 + m_1m_2n_3 \\ p &= n_1n_2m_3 - m_1n_2n_3, & q &= m_1n_2m_3 + n_1n_2n_3 \end{aligned}\right\} \qquad (5.5)$$

In the general case of the n-dimensional proper orthogonal group it is possible to factorize the general matrix in terms of the generalized Euler angles by standard methods (Murnaghan, 1962). For each particular matrix in the decomposition a correspondence similar to that in (5.1) and (5.2) can be obtained between the Cayley parameters and Euler angles.

According to the geometrical interpretation of the pure Lorentz transformations given by (3.4.3), if we take in (3.4.2) $n = p = q = s = t = 0$, the matrix

$$A = \frac{1}{m^2 - r^2}\begin{Bmatrix} m^2 + r^2 & 0 & 0 & -2mr \\ 0 & m^2 - r^2 & 0 & 0 \\ 0 & 0 & m^2 - r^2 & 0 \\ -2mr & 0 & 0 & m^2 + r^2 \end{Bmatrix} \qquad (5.6)$$

with $m^2 - r^2 > 0$, represents a pure Lorentz transformation along the x_1-axis. If the same transformation is written in terms of the hyperbolic functions, we conclude that

$$\operatorname{ch}\theta = \frac{m^2 + r^2}{m^2 - r^2}, \qquad \operatorname{sh}\theta = -\frac{2mr}{m^2 - r^2} \qquad (5.7)$$

with $-\infty < \theta < \infty$, and $0 \leqslant |r| < |m| < \infty$.

The general matrix of the proper Lorentz group can be factorized in the form of proper rotations and pure Lorentz transformations (Naimark, 1964b).

More explicitly, if $A(m, n, p, q)$ represents a proper rotation and $B(m, r, s, t)$ a pure Lorentz transformation, the general matrix of the proper Lorentz group can be expressed as follows

$$A(m,n,p,q,r,s,t,\lambda) = A(m_1,n_1,p_1,q_1)B(m_2,r_2,0,0)A(m_3,n_3,p_3,q_3)$$

The relation between these parameters can easily be obtained with the help of the multiplication law (4.1), namely

$$\left.\begin{aligned}
m &= m_1m_2m_3 - n_1m_2n_3 - p_1r_2p_3 - q_1r_2q_3,\\
n &= n_1m_2m_3 + m_1m_2n_3 + q_1r_2p_3 - p_1r_2q_3,\\
p &= p_1r_2m_3 - q_1r_2n_3 + m_1m_2p_3 + n_1m_2q_3,\\
q &= q_1r_2m_3 + p_1r_2n_3 - n_1m_2p_3 + m_1m_2q_3,\\
r &= m_1r_2m_3 + n_1r_2n_3 + p_1r_2p_3 - q_1r_2q_3,\\
s &= n_1r_2m_3 - m_1r_2n_3 - q_1r_2p_3 - p_1r_2q_3\\
t &= -p_1r_2m_3 - q_1r_2n_3 + m_1r_2p_3 - n_1r_2q_3\\
\lambda &= q_1r_2m_3 - p_1r_2n_3 + n_1r_2p_3 + m_1r_2q_3
\end{aligned}\right\} \quad (5.8)$$

In order to factorize the $SU(2)$ group, in terms of Cayley parameters, we take expression (3.2.4) and make alternatively 2 parameters equal to zero. We obtain

$$A = \frac{1}{l^2+a^2}\begin{pmatrix} l^2-a^2-i2la & 0 \\ 0 & l^2-a^2+i2la \end{pmatrix}, \qquad r=s=0 \qquad (5.9)$$

$$A = \frac{1}{l^2+r^2}\begin{pmatrix} l^2-r^2 & -2lr \\ 2lr & l^2-r^2 \end{pmatrix}, \qquad a=s=0 \qquad (5.10)$$

$$A = \frac{1}{l^2+r^2}\begin{pmatrix} l^2-s^2 & -i2ls \\ -i2ls & l^2-s^2 \end{pmatrix}, \qquad a=r=0 \qquad (5.11)$$

From the homomorphism between the $SO(3)$ and $SU(2)$ groups and its parametrization (3.1.2) and (3.1.3), we deduce that (5.9) corresponds to a 3-dimensional orthogonal matrix with

$$\frac{m}{\sqrt{(m^2+n^2)}} = \frac{l^2-a^2}{l^2+a^2} = \cos\frac{\phi}{2}, \qquad \frac{n}{\sqrt{(m^2+n^2)}} = \frac{2la}{l^2+a^2} = \sin\frac{\phi}{2} \quad (5.9')$$

where ϕ is the angle of rotation around the x_3-axis. (5.10) corresponds to a matrix of rotation with

$$\frac{m}{\sqrt{(m^2+p^2)}} = \frac{l^2-r^2}{l^2+r^2} = \cos\frac{\theta}{2}, \qquad \frac{p}{\sqrt{(m^2+p^2)}} = \frac{2lr}{l^2+r^2} = \sin\frac{\theta}{2} \quad (5.10')$$

where θ is the angle of rotation around the x_2-axis. (5.11) corresponds to a matrix of rotation with

$$\frac{m}{\sqrt{(m^2+q^2)}} = \frac{l^2-s^2}{l^2+s^2} = \cos\frac{\psi}{2}, \qquad \frac{q}{\sqrt{(m^2+q^2)}} = \frac{2ls}{l^2+s^2} = \sin\frac{\psi}{2} \quad (5.11')$$

where ψ is the angle of rotation around the x_1-axis.

In analogy with the Euler factorization of the $SU(2)$ group (Wigner, 1959b), we obtain

$$A = \frac{1}{\Delta}\begin{pmatrix} l_1^2 - a_1^2 - i2l_1a_1 & 0 \\ 0 & l_1^2 - a_1^2 + i2l_1a_1 \end{pmatrix}\begin{pmatrix} l_2^2 - a_2^2 & -2l_2a_2 \\ 2l_2a_2 & l_2^2 - a_2^2 \end{pmatrix}$$

$$\times \begin{pmatrix} l_3^2 - a_3^2 - i2l_3a_3 & 0 \\ 0 & l_3^2 - a_3^2 + i2l_3a_3 \end{pmatrix}$$

with

$$\Delta = (l_1^2 + a_1^2)(l_2^2 + a_2^2)(l_3^2 + a_3^2) \quad (5.12)$$

Symbolically,

$$A(l, a, r, s) = A(l_1, a_1, 0, 0)A(l_2, 0, a_2, 0)A(l_3, a_3, 0, 0) \quad (5.13)$$

where the parameters satisfy

$$\left.\begin{array}{ll} l = l_1l_2l_3 - a_1l_2a_3, & a = a_1l_2l_3 + l_1l_2a_3 \\ r = a_1a_2l_3 - l_1a_2a_3, & q = l_1a_2l_3 + a_1a_2a_3 \end{array}\right\} \quad (5.14)$$

In the case of the $SU(3)$ parametrization each parameter in the off-diagonal of (3.3.1) gives rise to different traceless matrices S,

$$S = \frac{1}{l_1}\begin{pmatrix} 0 & r_1 & 0 \\ -r_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \frac{1}{l_2}\begin{pmatrix} 0 & ir_2 & 0 \\ ir_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{etc.} \quad (5.15)$$

and the corresponding matrices A, given by (2.5) are

$$A = \frac{1}{l_1^2 + r_1^2}\begin{pmatrix} l_1^2 - r_1^2 & -2l_1r_1 & 0 \\ 2l_1r_1 & l_1^2 - r_1^2 & 0 \\ 0 & 0 & l_1^2 + r_2^2 \end{pmatrix},$$

$$A = \frac{1}{l_2^2 + r_2^2}\begin{pmatrix} l_2^2 - r_2^2 & -i2l_2r_2 & 0 \\ -i2l_2r_2 & l_2^2 - r_2^2 & 0 \\ 0 & 0 & \end{pmatrix}, \qquad \text{etc.} \quad (5.16)$$

In these cases, we have

$$\det S = \operatorname{Tr} S = 0$$

and from (3.3.2) the unimodularity of A is automatically fulfilled.

When we take one parameter in the diagonal of S, say $a \neq 0$, and $b = c = 0$, the unimodularity condition (3.3.2) requires $a = 0$. In order to obtain non-zero but traceless matrices S, we choose the one-parameter matrices as follows

$$S = \frac{1}{l}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \frac{1}{l}\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -b \end{pmatrix},$$

$$S = \frac{1}{l}\begin{pmatrix} -c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \tag{5.17}$$

and from (2.5)

$$A = \frac{1}{l^2 + a^2}\begin{pmatrix} (l + ia)^2 & 0 & 0 \\ 0 & (l - ia)^2 & 0 \\ 0 & 0 & l^2 + a^2 \end{pmatrix} \quad \text{etc.} \tag{5.18}$$

The general matrix of the group $SU(3)$ is factorized with the help of (5.16) and (5.18) following the standard method (Murnaghan, 1962).

It can be seen that the first two matrices S given by (5.15) and the first of (5.17), after dividing by the corresponding parameters, are identical to the canonical generators of the $SU(3)$ group

$$E_\alpha + E_{-\alpha}, \qquad E_\alpha - E_{-\alpha}, \qquad H_\alpha$$

where α is the root $(1, -1, 0)$. This result, common to all semisimple groups, is not surprising, because the matrices S of the Cayley parametrization and the infinitesimal generators of the semisimple groups satisfy the same linear conditions given in Table 1, and therefore both satisfy the same commutation relations.

6. *The Hypothesis of a* (3 + 1)*-Dimensional Cubic Lattice*

The transformations we have described so far are acting on a linear vector space defined over the field of real or complex numbers. We want to restrict the field of this vector space to the field of rational and integer numbers. This means that the components of the vectors are no longer continuous variables. Instead, the components will only be discrete variables and the functions connecting these variables will take rational values.

In particular, this hypothesis applied to the 4-dimensional Minkowski space is expressed by the following assumptions:

(1) The space-time variables can take only integer values. This is understood on the basis of a 3-dimensional spacial cubic lattice supplemented with a 1-dimensional temporal chain, in which the events can only be assigned to one of the points of this (3 + 1)-dimensional cubic lattice.
(2) Any magnitude derived from the space-time variables should also be integer or at least rational, and the laws of physics connecting these magnitudes should be described by functions of rational character, in the following sense: the dependent as well as independent variables should take only integer or rational values. Strictly speaking this assumption does not follow from the first, although it is consistent with it.
(3) Since the 'edge' of the basic space-time cubic lattice is assumed to be very small, the laws of physics will 'appear' as continuous for big values of these variables. The functions describing the laws of physics will be the same, but in the case of discrete space-time variables, only the rational points of these functions will be taken as possible values, while in the continuous case the irrational points are also accepted. This correspondence law does not mean that we have to take the limit in going from the discrete case to the continuous case. Instead we start from the continuous space-time, which is the domain for the classic and quantum physics, and assume that the equation of motion are still valid in the discrete space-time, but restricted only to the integer or rational values of their variables.

According to special relativity all inertial systems must be equivalent. Therefore, the assumptions above stated must be valid for any arbitrary inertial system. It follows that the rational character of the laws of motion in a particular system must be preserved under a proper Lorentz transformation, and that the space-time variables should be integer in any arbitrary inertial system. These two conservation laws will impose very strong conditions in the transformations connecting different inertial systems.

In the following, we will study how to obtain the rational points of the simplest and most often used transcendental functions, and the particular linear transformations which preserve this rationality condition. Then we will try to apply these results to some simple equation of motion in relativistic physics.

7. *Rationalization of Elementary Transcendental Functions*

7.1. *The Trigonometric Functions*

The solutions of the diophantine equation $x^2 + y^2 = r^2$ are given by (Sierpiński, 1964a)

$$x = (m^2 - n^2)d, \qquad y = 2mnd, \qquad r = (m^2 + n^2)d \tag{7.1.1}$$

where $m, n < m$ and d are integer numbers, provided the solutions for x, y are interchanged and added to them. If $z = m + in$ is a complex number with integer components (7.1.1) can be written

$$x = d \operatorname{Re} z^2, \qquad y = d \operatorname{Im} z^2, \qquad r = d|z|^2 \tag{7.1.1'}$$

From the definition of the trigonometric functions $\cos\alpha = x/r$, $\sin\alpha = y/r$, and from the completeness of the solutions (7.1.1) it follows that the rational values of these functions are exhausted by

$$\cos\alpha = \frac{m^2 - n^2}{m^2 + n^2} = \frac{\operatorname{Re} z^2}{|z|^2}, \qquad \sin\alpha = \frac{2mn}{m^2 + n^2} = \frac{\operatorname{Im} z^2}{|z|^2} \tag{7.1.2}$$

and by the same expressions for cos α and sin α interchanged.

Obviously

$$e^{i\alpha} = \frac{(m + in)^2}{m^2 + n^2} = \frac{z^2}{|z|^2} \tag{7.1.3}$$

and for integer value of κ

$$e^{i\kappa\alpha} = \frac{z^{2\kappa}}{|z|^{2\kappa}} = \cos\kappa\alpha + i\sin\kappa\alpha \tag{7.1.4}$$

hence

$$\cos\kappa\alpha = \frac{1}{2}\left\{\frac{z^{2\kappa}}{|z|^{2\kappa}} + \frac{z^{*2\kappa}}{|z|^{2\kappa}}\right\}, \qquad \sin\kappa\alpha = \frac{1}{2i}\left\{\frac{z^{2\kappa}}{|z|^{2\kappa}} - \frac{z^{*2\kappa}}{|z|^{2\kappa}}\right\} \tag{7.1.5}$$

These rationalized functions are non-continuous but they satisfy the functional equations

$$\left.\begin{aligned}
&\cos(\kappa_1 + \kappa_2)\alpha = \cos\kappa_1\alpha\cos\kappa_2\alpha - \sin\kappa_1\alpha\sin\kappa_2\alpha \\
&\sin(\kappa_1 + \kappa_2)\alpha = \sin\kappa_1\alpha\cos\kappa_2\alpha + \sin\kappa_2\alpha\cos\kappa_1\alpha \\
&\cos^2\kappa\alpha + \sin^2\kappa\alpha = 1
\end{aligned}\right\} \tag{7.1.6}$$

The argument α in (7.1.2) and (7.1.5) still remains irrational since it represents twice the area of the circular sector whose angle is α in a circle of unit radius. In order to rationalize it we adopt the convention that α is equal to twice the area bounded by two radii and the joining chord divided by the square of the radius. To be more explicit, suppose we construct a set of points P_κ, whose components are the real and imaginary part of

$$Z_\kappa = z^{2\kappa}|z|^{2(p-\kappa)}, \qquad \kappa = 0, 1, 2, \ldots, p$$

where $z = m + in$, m, n, p, κ positive integers. The distance of all these points to the origin is always $|z|^{2p}$. We define the angle between the radii of two consecutive points P_κ and $P_{\kappa+1}$ as twice the area of the triangle $OP_\kappa P_{\kappa+1}$ divided by the square of the radius OP_κ. Obviously this angle is the same for arbitrary P_κ, although it depends on the vector $z = m + in$. So the argument of the trigonometric functions as defined becomes a rational number, namely

$$\alpha = \frac{2mn}{m^2 + n^2} = \operatorname{sen}\alpha \qquad (m, n \text{ integers}) \tag{7.1.7}$$

7.2. *The Hyperbolic Functions*

The solutions of the diophantine equation $x^2 - y^2 = r^2$ are given by

$$x = (m^2 + n^2)d, \qquad y = 2mnd, \qquad r = (m^2 - n^2)d \tag{7.2.1}$$

where m, $|n| < |m|$, and d are integers. In order to give a geometric interpretation to this solution we take from (4.3) the particular hypercomplex number

$$u = m + nu_4 \qquad (u_4^2 = 1)$$

and define

$$\left.\begin{aligned} u^* &= m - nu_4 \\ |u|^2 &= uu^* = m^2 + n^2 \end{aligned}\right\} \tag{7.2.2}$$

hence (7.2.1) can be written

$$x = \frac{u^2 + u^{*2}}{2}, \qquad y = \frac{u^2 - u^{*2}}{2} u_4, \qquad r = |u|^2 \tag{7.2.3}$$

The hyperbolic functions can be defined over the hyperbola $x^2 - y^2 = r^2$ as $\text{ch}\,\beta = x/r$, $\text{sh}\,\beta = y/r$, where $\beta/2$ is the area bounded by the x-axis, the radius vector of the point (x, y) and the hyperbola divided by r^2. Then, from (7.2.1) the rational values of the hyperbolic functions are

$$\text{ch}\,\beta = \frac{m^2 + n^2}{m^2 - n^2}, \qquad \text{sh}\,\beta = \frac{2mn}{m^2 - n^2}, \qquad |n| < |m| \tag{7.2.4}$$

hence

$$\left.\begin{aligned} e^{\beta} &= \text{ch}\,\beta + \text{sh}\,\beta = \frac{m+n}{m-n}, \qquad |n| < |m| \\ e^{-\beta} &= \text{ch}\,\beta - \text{sh}\,\beta = \frac{m-n}{m+n} \end{aligned}\right\} \tag{7.2.5}$$

gives also the rational values of the exponential function.

For an argument $\kappa\beta$, κ integer, (7.2.5) becomes

$$\left.\begin{aligned} \text{ch}\,\kappa\beta + \text{sh}\,\kappa\beta &= e^{\kappa\beta} = \left(\frac{m+n}{m-n}\right)^{\kappa} \qquad |n| < |m| \\ \text{ch}\,\kappa\beta - \text{sh}\,\kappa\beta &= e^{-\kappa\beta} = \left(\frac{m-n}{m+n}\right)^{\kappa} \end{aligned}\right\}$$

and from these expressions it is easy to prove

$$\operatorname{ch}\kappa\beta = \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}} + \frac{u^{*2\kappa}}{|u|^{2\kappa}}\right\}, \qquad \operatorname{sh}\kappa\beta = \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}} - \frac{u^{*2\kappa}}{|u|^{2}}\right\}u_4 \quad (7.2.6)$$

where $u = m + nu_4$ $(u_4^2 = 1)$, and m, n, κ are integers.

The rationalized function (7.2.6) satisfy

$$\begin{aligned}
&\operatorname{ch}(\kappa_1+\kappa_2)\beta = \operatorname{ch}\kappa_1\beta\operatorname{ch}\kappa_2\beta + \operatorname{sh}\kappa_1\beta\operatorname{sh}\kappa_2\beta\\
&\operatorname{sh}(\kappa_1+\kappa_2)\beta = \operatorname{sh}\kappa_1\beta\operatorname{ch}\kappa_2\beta + \operatorname{sh}\kappa_2\beta\operatorname{ch}\kappa_1\beta\\
&\operatorname{sh}^2\kappa\beta + \operatorname{ch}^2\kappa\beta = 1\\
&\operatorname{ch}\kappa\beta + u_4\operatorname{sh}\kappa\beta = \frac{u^{2\kappa}}{|u|^{2\kappa}} \equiv e^{u_4\kappa\beta}
\end{aligned} \qquad (7.2.7)$$

the last definition in analogy with (7.1.4).

If we take the set of points $P_\kappa(x, y)$ whose components are the components of the hyper-complex numbers

$$U_\kappa = u^{2\kappa}|u|^{2(p-\kappa)} \qquad (\kappa = 0, 1, 2, p) \qquad (7.2.8)$$

the points P_κ lie in the hyperbola $x^2 - y^2 = r^2, r = |u|^{2P}$, and for each point $\operatorname{ch}\kappa\beta = x/v$, $\operatorname{sh}\kappa\beta = y/r$ satisfy the expressions (7.2.6). We define the rationalized argument for the hyperbolic functions β as twice the area of the triangle $OP_\kappa P_{\kappa+1}$ divided by $(OP_\kappa)^2$ where P_κ and $P_{\kappa+1}$ are two consecutive points, therefore

$$\beta = \frac{2mn}{m^2 - n^2} = \operatorname{sh}\beta \qquad |n| < |m| \qquad (7.2.9)$$

This argument does not depend on the particular point P_κ, although it depends on the number $u = m + nu_4$.

The rationalized trigonometric functions (7.1.5) and the rationalized hyperbolic function (7.2.6) satisfy the difference equations

$$\left.\begin{aligned}
&\Delta^2\cos\kappa\alpha + 2(1-\cos\alpha)\cos(\kappa+1)\alpha = 0\\
&\Delta^2\sin\kappa\alpha + 2(1-\cos\alpha)\sin(\kappa+1)\alpha = 0\\
&\Delta^2\operatorname{ch}\kappa\beta + 2(1-\operatorname{ch}\beta)\operatorname{ch}(\kappa+1)\beta = 0\\
&\Delta^2\operatorname{sh}\kappa\beta + 2(1-\operatorname{ch}\beta)\operatorname{sh}(\kappa+1)\beta = 0
\end{aligned}\right\} \qquad (7.2.10)$$

where the difference operator is defined as usual

$$\Delta f(\kappa) = f(\kappa+1) - f(\kappa)$$

and α and β satisfy (7.1.7) and (7.2.9) respectively.

From (7.1.4) and (7.2.9) and (7.2.10) we also obtain

$$\Delta^2 \frac{z^{2\kappa}}{|z|^{2\kappa}} - \frac{(z-z^*)^2}{|z|^2} \cdot \frac{z^{2(\kappa+1)}}{|z|^{2(\kappa+1)}} = 0$$

$$\Delta^2 \frac{u^{2\kappa}}{|u|^{2\kappa}} - \frac{(u-u^*)^2}{|u|^2} \cdot \frac{u^{2(\kappa+1)}}{|u|^{2(\kappa+1)}} = 0$$

7.3. *Generalized Trigonometric Functions*

Let C be a hypercircle of unit radius in a 4-dimensional space, defined by the intersection of the hypersphere

$$x^2 + y^2 + z^2 + t^2 = 1 \tag{7.3.1}$$

with one hyperplane containing the vector (0, 0, 0, 1). We can define generalized trigonometric functions on this circle by

$$\sin_1 \alpha = x, \qquad \sin_2 \alpha = y, \qquad \sin_3 \alpha = z, \qquad \cos \alpha = t \tag{7.3.2}$$

where α is twice the area of the sector bounded by the t-axis and the radius of the point (x, y, z, t).

These functions can be rationalized with the help of the hypercomplex numbers

$$\left.\begin{aligned} \omega &= m + nu_1 + pu_2 + qu_3 \\ \omega^* &= m - nu_1 - pu_2 - qu_3 \end{aligned}\right\} \qquad |\omega|^2 = \omega\omega^* \tag{7.3.3}$$

where u_1, u_2 and u_3 satisfy the multiplication law of Table 2. In a similar way as used for the trigonometric functions we have

$$\begin{aligned} \cos \kappa\alpha &= \frac{1}{2}\left\{\frac{\omega^{2\kappa}}{|\omega|^{2\kappa}} + \frac{\omega^{*2\kappa}}{|\omega|^{2\kappa}}\right\} \\ \sin_1 \kappa\alpha &= \frac{1}{2}\left\{\frac{\omega^{2\kappa}}{|\omega|^{2\kappa}} u_1 - u_1 \frac{\omega^{*2\kappa}}{|\omega|^{2\kappa}}\right\} \\ \sin_2 \kappa\alpha &= \frac{1}{2}\left\{\frac{\omega^{2\kappa}}{|\omega|^{2\kappa}} u_2 - u_2 \frac{\omega^{*2\kappa}}{|\omega|^{2\kappa}}\right\} \\ \sin_3 \kappa\alpha &= \frac{1}{2}\left\{\frac{\omega^{2\kappa}}{|\omega|^{2\kappa}} u_3 - u_3 \frac{\omega^{*2\kappa}}{|\omega|^{2\kappa}}\right\} \end{aligned} \tag{7.3.4}$$

where m, n, p, q, κ can take only integer values. These functions depend on the particular hypercomplex number ω, and they satisfy the functional equations, for κ, l integers,

$$\cos(\kappa+l)\alpha = \cos\kappa\alpha\cos l\alpha - \sin_1\kappa\alpha\sin_1 l\alpha - \sin_2\kappa\alpha \sin_2 l\alpha - \sin_3\kappa\alpha\sin_3 l\alpha$$
$$\sin_1(\kappa+l)\alpha = \cos\kappa\alpha\sin_1 l\alpha + \sin_1\kappa\alpha\cos l\alpha$$
$$\sin_2(\kappa+l)\alpha = \cos\kappa\alpha\sin_2 l\alpha + \sin_2\kappa\alpha\cos l\alpha$$
$$\sin_3(\kappa+l)\alpha = \cos\kappa\alpha\sin_3 l\alpha + \sin_3\kappa\alpha\cos l\alpha$$
$$\cos^2\kappa\alpha + \sin_1^2\kappa\alpha + \sin_2^2\kappa\alpha + \sin_3^2\kappa\alpha = 1 \tag{7.3.5}$$

Notice that the basis elements with negative sign $(-u_1, -u_2, -u_3)$ satisfy the same algebra of the basis elements i, j, κ for the quaternions. With this substitution (7.3.4) can be expressed also in the field of quaternions. In particular, for $\kappa = 1$, it holds

$$\cos\alpha = \frac{m^2-n^2-p^2-q^2}{m^2+n^2+p^2+q^2}, \qquad \sin_1\alpha = \frac{2mn}{m^2+n^2+p^2+q^2}$$

$$\sin_2\alpha = \frac{2mp}{m^2+n^2+p^2+q^2}, \qquad \sin_3\alpha = \frac{2mq}{m^2+n^2+p^2+q^2} \tag{7.3.6}$$

The functions (7.3.4) also satisfy the following difference equations:

$$\Delta^2\cos\kappa\alpha + 2(1-\cos\alpha)\cos(\kappa+1)\alpha = 0$$
$$\Delta^2\sin_1\kappa\alpha + 2(1-\cos\alpha)\sin_1(\kappa+1)\alpha = 0$$
$$\Delta^2\sin_2\kappa\alpha + 2(1-\cos\alpha)\sin_2(\kappa+1)\alpha = 0$$
$$\Delta^2\sin_3\kappa\alpha + 2(1-\cos\alpha)\sin_3(\kappa+1)\alpha = 0 \tag{7.3.7}$$

Since $\cos\kappa\alpha + u_1\sin_1\kappa\alpha + u_2\sin_2\kappa\alpha + u_3\sin_3\kappa\alpha = \omega^{2\kappa}/|\omega|^{2\kappa}$ we have from (7.3.7)

$$\Delta^2\frac{\omega^{2\kappa}}{|\omega|^{2\kappa}} - \frac{(\omega-\omega^*)^2}{|\omega|^2}\frac{\omega^{2(\kappa+1)}}{|\omega|^{2(\kappa+1)}} = 0 \tag{7.3.7'}$$

7.4. *Generalized Hyperbolic Functions*

Let S be an hyperbola defined by the intersection of the hyperboloid

$$t^2 - x^2 - y^2 - z^2 = 1$$

and one plane containing the vector (1, 0, 0, 0). We can define generalized hyperbolic functions on this hyperbola by

$$\mathrm{ch}\,\beta = t, \qquad \mathrm{sh}_1\beta = x, \qquad \mathrm{sh}_2\beta = y, \qquad \mathrm{sh}_3\beta = z \tag{7.4.1}$$

where β is twice the area bounded by the hyperbole S, the t-axis and the radius vector of the point (t, x, y, z).

Using the hypercomplex numbers

$$\left.\begin{aligned} u &= m + ru_4 + su_5 + tu_6 \\ u^* &= m - ru_4 - su_5 - tu_6 \\ |u|^2 &= uu^* \end{aligned}\right\} \tag{7.4.2}$$

where u_4, u_5 and u_6 satisfy the multiplication law of Table 2, the rationalized functions (7.4.1) can be written

$$\begin{aligned} \operatorname{ch}\kappa\beta &= \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}} + \frac{u^{*2\kappa}}{|u|^{2\kappa}}\right\} \\ \operatorname{sh}_1\kappa\beta &= \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}}u_4 - u_4\frac{u^{*2\kappa}}{|u|^{2\kappa}}\right\} \\ \operatorname{sh}_2\kappa\beta &= \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}}u_5 - u_5\frac{u^{*2\kappa}}{|u|^{2\kappa}}\right\} \\ \operatorname{sh}_3\kappa\beta &= \frac{1}{2}\left\{\frac{u^{2\kappa}}{|u|^{2\kappa}}u_6 - u_6\frac{u^{*2\kappa}}{|u|^{2\kappa}}\right\} \end{aligned} \tag{7.4.3}$$

where m, r, s, t, κ can take only integer values. Obviously these generalized hyperbolic functions depend on the chosen number (7.4.2); they satisfy the functional equations

$$\begin{aligned} &\operatorname{ch}(\kappa + l)\beta = \operatorname{ch}\kappa\beta\operatorname{ch}l\beta + \operatorname{sh}_1\kappa\beta\operatorname{sh}_1 l\beta + \operatorname{sh}_2\kappa\beta\operatorname{sh}_2 l\beta + \operatorname{sh}_3\kappa\beta\operatorname{sh}_3 l\beta \\ &\operatorname{sh}_1(\kappa + l)\beta = \operatorname{ch}\kappa\beta\operatorname{sh}_1 l\beta + \operatorname{sh}_1\kappa\beta\operatorname{ch}l\beta \\ &\operatorname{sh}_2(\kappa + l)\beta = \operatorname{ch}\kappa\beta\operatorname{sh}_2 l\beta + \operatorname{sh}_2\kappa\beta\operatorname{ch}l\beta \\ &\operatorname{sh}_3(\kappa + l)\beta = \operatorname{ch}\kappa\beta\operatorname{sh}_3 l\beta + \operatorname{sh}_3\kappa\beta\operatorname{ch}l\beta \\ &\operatorname{ch}^2\kappa\beta - \operatorname{sh}_1^2\kappa\beta - \operatorname{sh}_2^2\kappa\beta - \operatorname{sh}_3^2\kappa\beta = 1 \end{aligned} \tag{7.4.4}$$

as well as the difference equations

$$\left.\begin{aligned} &\Delta^2\operatorname{ch}\kappa\beta + 2(1 - \operatorname{ch}\beta)\operatorname{ch}(\kappa + 1)\beta = 0 \\ &\Delta^2\operatorname{sh}_1\kappa\beta + 2(1 - \operatorname{ch}\beta)\operatorname{sh}_1(\kappa + 1)\beta = 0 \\ &\Delta^2\operatorname{sh}_2\kappa\beta + 2(1 - \operatorname{ch}\beta)\operatorname{sh}_2(\kappa + 1)\beta = 0 \\ &\Delta^2\operatorname{sh}_3\kappa\beta + 2(1 - \operatorname{ch}\beta)\operatorname{sh}_3(\kappa + 1)\beta = 0 \end{aligned}\right\} \tag{7.4.5}$$

Since

$$\operatorname{ch}\kappa\beta + u_4\operatorname{sh}_1\kappa\beta + u_5\operatorname{sh}_2\kappa\beta + u_6\operatorname{sh}_3\kappa\beta = \frac{u^{2\kappa}}{|u|^{2\kappa}}$$

we have from (7.4.5)

$$\Delta^2 \frac{u^{2\kappa}}{|u|^{2\kappa}} - \frac{(u-u^*)^2}{|u|^2} \frac{u^{2(\kappa+1)}}{|u|^{2(\kappa+1)}} = 0$$

8. *Other Rationalized Elementary Functions*

8.1. *Rational Points of Quadratic Equations*

The Cayley parametrization of the semisimple groups gives a simple method to find the integral solutions of quadratic forms. Let $G_{j\kappa}x_j^* x_\kappa$ be a non-degenerate bilinear expression which is left invariant under a semisimple Lie group $\mathscr{A}$ and let A be the general element of the group $\mathscr{A}$ expressed in terms of Cayley parameters. If we impose on these parameters the condition of being integers, from (2.2) it can be seen that each column, say i, of the matrix A gives a set of rational points that satisfy the quadratic equation $G_{j\kappa}A^*_{ji}A_{\kappa i} = G_{ii}$.

Take, for instance, the group $SO(3)$ and the Cayley parametrization given by (3.1.2). The elements of the last column,

$$A_{13} = \frac{2mn+2nq}{m^2+n^2+p^2+q^2}, \qquad A_{23} = \frac{-2mq+2np}{m^2+n^2+p^2+q^2},$$

$$A_{33} = \frac{m^2+n^2-p^2-q^2}{m^2+n^2+p^2+q^2} \tag{8.1.1}$$

with m, n, p, q integers are the rational points of the expression $A_{13}^2 + A_{23}^2 + A_{33}^2 = 1$.

If we define

$$x = d(2mp+2nq), \qquad y = d(-2mq+2np)$$

$$z = d(m^2+n^2-p^2-q^2), \qquad r = d(m^2+n^2+p^2+q^2) \tag{8.1.2}$$

from (8.1.1) it follows that (8.1.2) gives the integral solutions of $x^2+y^2+z^2=r^2$, a result that was derived first by Carmichael (1915).

The rational values of trigonometric and hyperbolic functions can also be derived with this method using the Cayley parametrization of the groups $SO(2)$ and $SO(1, 1)$ respectively.

In the case of the Lorentz group, each column in (3.4.2) gives a set of rational points of $A_{1\mu}^2 + A_{2\mu}^2 + A_{3\mu}^2 - A_{4\mu}^2 = 1$ ($\mu = 1, 2, 3, 4$). From these expressions it follows that, if we take the last column

$$\left.\begin{aligned}
x &= -2mr(m^2+q^2) + 2ms(mn-pq) - 2mt(mp+nq)\\
y &= -2mr(mn+pq) - 2ms(m^2+p^2) + 2mt(mq-pn)\\
z &= 2mr(mr-nq) - 2ms(mq+np) - 2mt(m^2+n^2)\\
t &= m^2(m^2+n^2+p^2+q^2+r^2+s^2+t^2) + (nt+ps+qr)^2\\
r &= m^2(m^2+n^2+p^2+q^2-r^2-s^2-t^2) - (nt+ps+qr)^2
\end{aligned}\right\} \tag{8.1.3}$$

with m, n, p, q, r, s, t, integer numbers, we obtain the solutions of the diophantine equation

$$t^2 - x^2 - y^2 - z^2 = r^2$$

8.2. *Rational Periodic Functions*

Given a rational trigonometric function (7.1.5), is it possible to find a positive integer l, such that $\cos(\kappa + l)\alpha = \cos\kappa\alpha$ and $\sin(\kappa + l)\alpha = \sin\kappa\alpha$, in other words, to have a rational trigonometric function which is periodic with period l?

If we define

$$\phi(\kappa) = \frac{z^{2\kappa}}{|z|^{2\kappa}} = \cos\kappa\alpha + i\sin\kappa\alpha \tag{8.2.1}$$

where $z = m + ni$, m, n integers, the periodicity condition reads

$$\phi(k + l) = \phi(\kappa) \qquad \text{or} \qquad z^{2l} = |z|^{2l} \tag{8.2.2}$$

One finds the following primitive solutions (which cannot be decomposed in the product of other solutions) of (8.2.2) for

$$\left.\begin{aligned} l = 1&: \phi_1(\kappa) = 1^\kappa \\ l = 2&: \phi_2(\kappa) = (-1)^\kappa \\ l = 4&: \phi_4(\kappa) = i^\kappa \end{aligned}\right\} \tag{8.2.3}$$

For $l \neq 1, 2, 4$ *there is no solution of* (8.2.2):

Proof. The imaginary part of (8.2.2) reads, for odd l,

$$l(m^2 - n^2)^{l-1} 2mn - \binom{l}{3}(m^2 - n^2)^{l-3}(2mn)^3 + \cdots \pm (2mn)^l = 0$$

We can simplify this expression dividing by $2mn$ (which eliminates the trivial solution $m = 0$, or $n = 0$). Dividing again by $n^{2(l-1)}$ we obtain

$$l\left(\frac{m^2}{n^2} - 1\right)^{l-1} - \binom{l}{3}\left(\frac{m^2}{n^2} - 1\right)^{l-3} 4\frac{m^2}{n^2} + \cdots \pm \left(\frac{4m^2}{n^2}\right)^{(l-1)/2} = 0 \tag{8.2.4}$$

This is a diophantine equation of the variable m^2/n^2. By congruence considerations (Sierpiński, 1964b), the solutions, if existing, is a rational number whose numerator must be a divisor of l, and the denominator must also be a divisor of l. Now l can be decomposed in a product of prime numbers. It is obvious that if (8.2.2) is a solution for given l, it will also be a solution for each of its prime components. Therefore we will find first the solution of (8.2.2) for l prime, and then try as possible solutions all the products constructed with these prime numbers.

For $l = 2$, we immediately get $\phi_2(\kappa)$ in (8.2.3). For $l > 2$ we have to solve

(8.2.4), because any prime $l > 2$ is odd. By what has been said above the only possible solutions of (8.2.4) with l prime, are $m^2/n^2 = 1, l, 1/l$, or

$$m^2 = n^2, \qquad m^2 = ln^2, \qquad lm^2 = n^2$$

The second and third choice are impossible with m, n integers, and the first choice does not satisfy the real part of (8.2.2), for l prime. Finally it can be checked that the only possible products constructed with $l = 2$, satisfying (8.2.2) are $l = 2 \times 2$, which correspond to $\phi_4(\kappa)$ in (8.2.3).

We can also construct rational periodic functions out of the generalized trigonometric functions (7.3.4). We define

$$\psi(\kappa) = \frac{\omega^{2\kappa}}{|\omega|^{2\kappa}}, \qquad \omega = m + nu_1 + pu_2 + qu_3 \tag{8.2.5}$$

where m, n, p, q are integers and u_1, u_2, u_3 satisfy the multiplication law of Table 2. The periodicity condition reads

$$\psi(\kappa + l) = \psi(\kappa) \qquad \text{or} \qquad \omega^{2l} = |\omega|^{2l} \tag{8.2.6}$$

for some positive integer l. One finds the primitive solutions of (8.2.6) for

$$l = 1: \psi_1(\kappa) = 1^\kappa$$

$$l = 2: \psi_2(\kappa) = (-1)^\kappa$$

$$\left.\begin{aligned} l = 3:\ \psi_3(\kappa) &= \left(\frac{-m + nu_1 + pu_2 + qu_3}{2m}\right)^\kappa, & 3m^2 &= n^2 + p^2 + q^2 \\ l = 4:\ \psi_4(\kappa) &= \left(\frac{nu_1 + pu_2 + qu_3}{m}\right)^\kappa, & m^2 &= n^2 + p^2 + q^2 \\ l = 6:\ \psi_6(\kappa) &= \left(\frac{m + 3nu_1 + 3pu_2 + 3qu_3}{2m}\right)^\kappa, & m^2 &= 3(n^2 + p^2 + q^2) \end{aligned}\right\} \tag{8.2.7}$$

From (7.3.4) and (8.2.7) we can deduce the rational trigonometric functions which are periodic. For $l = 3, 4, 6$ it can be seen that there exist infinite many solutions which satisfy the conditions for m, n, p, q.

If $l \neq 1, 2, 3, 4, 6$ there are no solutions of (8.2.6):

Proof. From the automorphism between the multiplicative group of quaternions and the proper rotation group, (8.2.6) can be expressed in terms of matrices (3.1.2), namely $A^l = 1$. If ϕ is the angle of rotation, this is equivalent to $\cos l\phi = 1$, $\sin l\phi = 0$, or,

$$(\cos\phi + i \sin\phi)^l = 1 \tag{8.2.8}$$

where l is a positive integer, and $\cos\phi$ is given by (3.1.4). As before, we only try the solutions for l prime. For $l = 2$ we obtain $\psi_2(\kappa) = (-1)^\kappa$. For $l > 2$, l is odd and the imaginary part of (8.2.8) reads

$$l(m^2 - n^2 - p^2 - q^2)^{l-1} - \binom{l}{3}(m^2 - n^2 - p^2 - q^2)^{l-3}4m^2(n^2 + p^2 + q^2)$$
$$+ \cdots \pm (4m^2(n^2 + p^2 + q^2))^{(l-1)/2} = 0 \quad (8.2.9)$$

By the same argument as before, the solutions of this diophantine equation must be of the form

$$m^2 = n^2 + p^2 + q^2, \qquad m^2 = l(n^2 + p^2 + q^2), \qquad lm^2 = n^2 + p^2 + q^2$$

The first choice does not satisfy the real part of (8.2.8). Substituting the second choice in (8.2.9), we get, after simplification,

$$l(l-1)^{l-1} - \binom{l}{3}(l-1)^{l-3}4l + \cdots + (4l)^{(l-1)/2} = 0 \quad (8.2.10)$$

If $l = 3$, we get $\psi_3(\kappa)$ of (8.2.7). If $l \neq 3$, each term in (8.2.10) can be decomposed as a product of prime numbers. The prime number l appears once in the first term, but it appears at least twice in the following terms (in the second term, $\binom{l}{3}$ is a multiple of l, because l is a prime number). Therefore the second choice is not possible for congruence considerations. The same can be proved for third choice.

Finally, from $l = 2$, and $l = 3$, the only possible combinations which gives a solution of (8.2.8) is $l = 2 \times 2$, and $l = 2 \times 3$, i.e. $\psi_4(\kappa)$ and $\psi_6(\kappa)$ respectively in (8.2.7).

8.3. *Integral Hyperbolic Functions*

A problem similar to the one in the last subsection arises with respect to the hyperbolic functions. Are there any particular values of m, r, s, t in (7.4.3) that gives integer values for the generalized hyperbolic functions? The question is equivalent to asking whether the diophantine equation

$$m^2 - r^2 - s^2 - t^2 = 1 \quad (8.3.1)$$

has non-trivial solutions. In this case

$$|u|^2 = 1, \qquad u = m + ru_4 + su_5 + tu_6 \quad (8.3.2)$$

and all the generalized hyperbolic functions (7.4.3) give automatically integer values. For the case of hyperbolic functions (7.2.6) ($s = t = 0$), the only solution of (8.3.1) is the trivial one, $r = 0, m = \pm 1$. In the case of generalized hyperbolic functions, (8.3.1) has infinitely many solutions, that can be constructed as follows: given an arbitrary m ($m \neq 0$), $m^2 - 1$ is a non-negative integer N, which can be always expressed as the sum of three squares (Sierpiński, 1964c)

$$N = m^2 - 1 = r^2 + s^2 + t^2$$

Then the numbers constructed with these values of m, r, s, t

$$u^{\kappa} = (m + ru_4 + su_5 + tu_6)^{\kappa} \tag{8.3.3}$$

and the functions derived from (8.3.3) and (7.4.3) have integer values for all κ.

It is interesting to observe that the set of all numbers u^{κ}, κ positive or negative integer, given by (8.3.3) form a group with respect to the multiplication law of the hypercomplex numbers (4.3). In particular, if

$$m = 2v_0^2 + 1, \qquad r = 2v_0v_1, \qquad s = 2v_0v_2, \qquad t = 2v_0v_3$$

where $v_1^2 + v_2^2 + v_3^2 - v_0^2 = 1$, then not only $|u|^2 = 1$ but also the vector difference between two consecutive u^{κ} and $u^{\kappa+1}$ possesses an integer magnitude. In fact

$$u^{\kappa+1} - u^{\kappa} = u^{\kappa}(u - 1) = u^{\kappa}(2v_0^2, 2v_0v_1, 2v_0v_2, 2v_0v_3)$$

and

$$|u^{\kappa+1} - u^{\kappa}| = 2v_0(v_1^2 + v_2^2 + v_3^2 - v_0^2)^{1/2} = 2v_0$$

hence this infinite regular 'polygon' has integer values for the length of its side and the components of its vortices.

9. *Semisimple Lie Groups with Rational and Integer Matrix Elements*

In the last paragraphs we have described a method to find the rational points of the trigonometric, hyperbolic and exponential functions. From these, it is very easy to construct the *rational linear transformations* that leave invariant some non-degenerate bi-linear form. By rational linear transformations, we understand those linear transformations whose coefficients can take only rational values. This can be done by two methods:

(a) Given a semisimple Lie group, we parametrize the defining representation using the standard decomposition in terms of the *Euler angles* (Murnaghan, 1962). It is known that all the matrix elements are constructed with the help of trigonometric, hyperbolic or exponential functions. If we take the rational points of these functions by the methods described in Section 7, we will obtain matrix elements of the corresponding group with rational values. As an example, take the Euler decomposition of the rotation group and substitute for the trigonometric functions their rational points given by (7.1.2). We will obtain the same result as (5.3) if we impose, in the last expression, the condition for m, n to be integer.

(b) The second method consists of the use of the *Cayley parametrization*, as was described in Sections 2 and 3, and then impose the condition on the parameters to be integers. Since all the matrix elements in Cayley parametrization are rational functions, their values will only be rational numbers, for integer values of the Cayley parameters. Methods (a) and (b) will lead to the same result, since the corresponding parameters are related to each other, as can be seen in the expressions (5.5) and (5.8) for the proper rotation group and proper Lorentz group.

The rationalization of the semisimple Lie groups makes possible the *conservation of the rational character* of the components of a vector. If we perform a transformation on this vector by some matrix with rational elements the transformed vector will also possess rational components.

Now we want to restrict more the conditions on the matrix elements of the semisimplex groups by requiring them to be not only rational but *integer numbers.* Doing this we obtain another useful property of linear transformations. In the hypothesis of N-dimensional cubic lattices, in which the coordinates take only integer values, any matrix with integer elements will transform a vector with integer components into another vector of the same character. We will study one example among the compact groups and another one among the non-compact groups.

(a) *The Proper Rotation Group.* If we impose in (3.1.2) all the matrix elements to be integers we must have either $m^2 - n^2 - p^2 + q^2 \geqslant m^2 + n^2 + p^2 + q^2$ or $m^2 - n^2 - p^2 + q^2 = 0$, and similar conditions for the rest of the diagonal elements. The first choice gives $n = p = q = 0$, which corresponds to the unit matrix. The second choice gives $m^2 = n^2 = p^2 = q^2$, which corresponds to the rotations

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{9.1}$$

and similar matrices with two arbitrary minus signs. The cubic lattice is left invariant under these rotations.

If we consider particular sublattice of the whole Euclidean lattice like the set of all points with coordinates $d(x, y, z), d, x, y, z$ integers, the rotation matrices for which $m^2 + n^2 + p^2 + q^2 = d$ will conserve the integral character of the points in this sublattice. Also the set of all points in the plane

$$Z_{\kappa l}^{(j)} = lz^{2\kappa} |z|^{2(j-\kappa)} \qquad (\kappa = 0, 1, 2, \ldots j)$$

with $z = m + in, j, \kappa, l, m, n$ integers, is also left invariant under a rotation of angle $\phi = \sin \phi = 2mn/(m^2 + n^2)$.

(b) *The Proper Lorentz Group.* A pure Lorentz transformation with $m^2 - r^2 - s^2 - t^2 = 1$ will be of the form

$$B(m, r, s, t) =$$

$$\begin{bmatrix} m^2 + r^2 - s^2 - t^2 & 2rs & 2rt & -2mr \\ 2rs & m^2 - r^2 + s^2 - t^2 & 2st & -2ms \\ 2rt & 2st & m^2 - r^2 - s^2 + t^2 & -2mt \\ -2mr & -2ms & -2mt & m^2 + r^2 + s^2 + t^2 \end{bmatrix} \tag{9.2}$$

where all matrix elements are integers. The solution of $m^2 - r^2 - s^2 - t^2 = 1$ can be obtained with the method explained in Subsection 8.3.

For the proper Lorentz group we can use the decomposition described in Section 5. The matrices $A(m, n, p, q)$ of the rotation group can only have integer elements if they are of the form (9.1) or some of the variations explained afterwards. Therefore the only matrix of the proper Lorentz group with integer elements will be

$$A(m_1, n_1, p_1, q_1) \quad B(m_2, r_2, s_2, t_2) \quad A(m_3, n_3, p_3, q_3) \qquad (9.2')$$

with $m_1^2 = n_1^2 = p_1^2 = q_1^2, m_2^2 - r_2^2 - s_2^2 - t_2^2 = 1, m_3^2 = n_3^2 = p_3^2 = q_3^2$.

10. *Simple Applications to Relativistic Mechanics in a Discrete Four-Dimensional Space*

We have now some mathematical tools to work out the description of physical laws in a discrete space-time world. Although the mathematical difficulties of this goal seem to be very great, because of the use of numerical analysis and difference equations, we will try some simple examples.

According to our first postulate (see Section 6) given a system of reference S, in a Minkowski space, the space-time variables $(\mathbf{x}, t)$ can take only integer values. We call this space (3 + 1)-dimensional cubic lattice. Every function of these variables $f(\mathbf{x}, t)$ will allow only integer values of its arguments. Therefore, the infinitesimal increments $(d\mathbf{x}, dt)$ should be changed by discrete intervals $(\Delta\mathbf{x}, \Delta t)$ with integer values. We have:

(a) Trajectory: $\mathbf{x} = \mathbf{x}(t)$ [10.1a]

(b) Velocity: $\mathbf{u} = \dfrac{\Delta\mathbf{x}}{\Delta t}$ [10.1b]

(c) Proper time: $\Delta\tau = \left(1 - \dfrac{u^2}{c^2}\right)^{1/2} \Delta t = \dfrac{1}{c}(c^2(\Delta t)^2 - (\Delta\mathbf{x})^2)^{1/2}$ [10.1c]

(d) Linear momentum: $\mathbf{p} = \dfrac{m_0 c\, \Delta\mathbf{x}}{(c^2(\Delta t)^2 - (\Delta\mathbf{x})^2)^{1/2}}$ [10.1d]

(e) Energy: $E = \dfrac{m_0 c^3\, \Delta t}{(c^2(\Delta t)^2 - (\Delta\mathbf{x})^2)^{1/2}}$ [10.1e]

According to our second postulate (Section 6) all functions involving the space-time variables must be rational, in the sense that their values must be rational numbers. For this reason, the expression $(c^2(\Delta t)^2 - (\Delta\mathbf{x})^2)^{1/2}$ which appears in (c), (d), (e) must be an integer, and this is true if the quantities $c\Delta t$, $\Delta\mathbf{x}$ simultaneously take some of the values for (t, x, y, z) given in formula (8.1.3).

The rest mass m_0 in (10.1d) and (10.1e) is independent of the space-time variables in the special relativity. The Lorentz invariant $(E^2/c^2) - \mathbf{p}^2 = m_0^2 c^2$ does not impose any constraint in m_0, because it is a common factor in both sides of this equation. However, its values should take integer values, with respect to some fundamental mass similarly to the space-time variables. This assumption, strictly speaking, is not contained in assumption (2) of Section 6, although is consistent with it.†

With these restrictions, the equation of motion of a particle in the system S is given by

$$\frac{\Delta \mathbf{p}}{\Delta t} = \mathbf{F}, \qquad \frac{\Delta E}{\Delta t} = \mathbf{F} \, . \, \mathbf{u} \tag{10.2}$$

where **p** and **u** are defined above and **F** is the force which is the cause of the change of the momentum. For a free particle, $\mathbf{F} = 0$, (10.2) gives $\Delta \mathbf{x}/\Delta t =$ const., but now $\Delta \mathbf{x}$ and Δt are constrained by the condition that $(c^2(\Delta t)^2 - (\Delta \mathbf{x})^2)^{1/2}$ should be integer.

Given a four vector x whose components are $(\mathbf{x}, ct)$ in an inertial system S, the same vector viewed from another inertial system S', will have components $(\mathbf{x}', ct')$ which are related to the old ones by $x' = Lx$, where L is a proper Lorentz transformation. It is easy to prove that the finite increments $(\Delta \mathbf{x}, c\Delta t)$ transform as a four vector, and so its 'length' $(c^2(\Delta t)^2 - (\Delta \mathbf{x})^2)^{1/2}$ is a Lorentz invariant. This shows immediately that the proper time in (10.1c) is a Lorentz invariant and the four-momentum $(\mathbf{p}, E/c^2)$ transforms as the four vector $(\Delta \mathbf{x}, c\Delta t)$. Besides that, if we impose the condition that $(c^2(\Delta t)^2 - (\Delta \mathbf{x})^2)^{1/2}$ be an integer, and that the Cayley parameters of the Lorentz transformation (3.4.2) take integer values, the rational character of the four-momentum will be preserved in any inertial system.

Further restriction will be necessary in order to preserve the integer values of the space-time variables. These restrictions can be satisfied if the Lorentz transformations are one of the matrices (9.2) or (9.2'). It means that the velocity of the system S' with respect to the system S, given by (3.4.3), is restricted by the condition $m^2 - r^2 - s^2 - t^2 = 1$.

Consider now the following examples in relativistic electrodynamics. The force on an electrically charged particle of charge e moving in a given electromagnetic field with velocity **u** relative to an inertial system S is given by (Møller, 1952a)

$$\mathbf{F} = e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{u} \times \mathbf{H})\right\} \tag{10.3}$$

which can be substituted in (10.2) to obtain the law of motion. The magnetic field due to a point charge moving with uniform velocity $\mathbf{u} = \Delta \mathbf{x}/\Delta t$ is (Moller, 1952b)

$$4\pi \mathbf{H} = \frac{e}{|r|^3} \frac{\Delta \mathbf{x} \times \mathbf{r}}{(c^2(\Delta t)^2 - (\Delta \mathbf{x})^2)^{1/2}} \tag{10.4}$$

† We are grateful to Professor Roman for this remark.

where **r** is the vector joining the point charge and the point where **H** is measured. In order to have a rational expression **H**, $(r_1^2 + r_2^2 + r_3^2)^{1/2}$ and $(c^2(\Delta t)^2 - (\Delta \mathrm{x})^2)^{1/2}$ should be an integer, which is obtained with the help of (8.1.2) and (8.1.3), and at the same time that e should be measured in natural units (electron charge = 1), and **H** should be measured in a system in which 4π does not appear in (10.4).

We assume that Maxwell's equation in an inertial system where the coordinates of a point event are integers should be written (Møller, 1952c)

$$\left.\begin{aligned} &\frac{\Delta_x H_x}{\Delta x} + \frac{\Delta_y H_y}{\Delta y} + \frac{\Delta_z H_z}{\Delta z} = 0 \\ &\frac{\Delta_y E_z}{\Delta y} - \frac{\Delta_z E_y}{\Delta z} + \frac{1}{c}\frac{\Delta_t H_x}{\Delta t} = 0, \qquad \text{etc.} \end{aligned}\right\} \tag{10.5a}$$

$$\left.\begin{aligned} &\frac{\Delta_x E_x}{\Delta x} + \frac{\Delta_y E_y}{\Delta y} + \frac{\Delta_z E_z}{\Delta z} = \rho \\ &\frac{\Delta_y H_z}{\Delta y} - \frac{\Delta_z H_y}{\Delta z} - \frac{1}{c}\frac{\Delta_t E_x}{\Delta t} = \frac{1}{c}\rho u_x, \qquad \text{etc.} \end{aligned}\right\} \tag{10.5b}$$

where **H**, **E**, ρ and **u** are functions of x, y, z, t. (ρ is the charge density and **u** the velocity with which the charges move in the inertial system S.)

We solve (10.5) in the particular case of a plane wave moving in the direction of the x-axis in vacuum ($\rho = 0$). Then the fields are functions of x, t only. By similar arguments to the continuous case (Møller, 1952d), and taking for simplicity $\Delta x = \Delta t = 1$, one can show that the functions H_y, H_z, E_y, E_z satisfy the wave equation

$$\Delta_{xx}^2 \phi(x, t) = c^2 \Delta_{tt}^2 \phi(x, t) \tag{10.6}$$

A solution of this equation will be the exponential function (or each of its components)

$$\phi(t + x) = \left(\frac{\omega^2}{|\omega|^2}\right)^{t+x} \tag{10.7}$$

which represents a plane wave going in the direction of the negative x-axis. (We take x, t in natural units, $c = 1$.)

In fact, from (7.3.7′) we have

$$\Delta_{xx}^2 \left(\frac{\omega^2}{|\omega|^2}\right)^{x+t} = \frac{(\omega - \omega^*)^2}{|\omega|^2} \left(\frac{\omega^2}{|\omega|^2}\right)^{x+t+1}$$

$$\Delta_{tt}^2 \left(\frac{\omega^2}{|\omega|^2}\right)^{x+t} = \left(\frac{\omega - \omega^*}{|\omega|^2}\right)^2 \left(\frac{\omega^2}{|\omega|^2}\right)^{x+t+1}$$

from which (10.7) follows.

The analog of (10.7) in the continuous case is the exponential function (or trigonometric functions)

$$e^{i2\pi(t+x)}, \qquad e^{i2\pi} = 1$$

Since this is a periodic function, we choose (10.7) to be one of the rational periodic functions given by (8.2.3) and (8.2.7), with period l, namely

$$\phi_l(t+x) = \left(\frac{\omega^2}{|\omega|^2}\right)^{(t+x)}, \qquad \left(\frac{\omega^2}{|\omega|^2}\right)^l = 1 \qquad (10.8)$$

Similar arguments can be made for the plane waves in the direction of the positive x-axis, $\phi(t-x)$, although we should work with the backwards difference operator.

From this simple example we see that assumption (3) of Section 6 is fulfilled, since the solution of the continuous case coincide in the rational points with the solution of the discrete case.

The period T of the solutions $\phi_l(t+x)$ must be a multiple of l, $T = jl$, $j = 1, 2, \ldots$, and the same for the wavelength $\lambda = jl$. (In the particular case of plane wave (10.8) $j = 1$.) As in the continuous case, we can define an angular frequency f and a wave number κ, as

$$f = \frac{l}{T} = \frac{1}{j}, \qquad \kappa = \frac{l}{\lambda} = \frac{1}{j}$$

and so our wave functions become

$$\phi_T(t+x) = \left(\frac{\omega^2}{|\omega|}\right)^{(1/j)(t+x)}, \qquad \left(\frac{\omega^2}{|\omega|^2}\right)^T = 1 \qquad (10.9)$$

In this case only those values of t or x, for which t/j or x/j are integer, have physical meaning.

If the plane wave is moving perpendicular to a different direction than to the x-axis the lattice structure requires that

$$T = jlr \qquad \text{and} \qquad \lambda = jlr$$

where $r = |\mathbf{r}|$ is a particular solution of the diophantine equation $r^2 = r_1^2 + r_2^2 + r_2^3$ and $\mathbf{r}(r_1 r_2 r_3)$ are the components of a vector $\mathbf{r}$ perpendicular to the plane wave. Calling

$$\kappa_0 = \frac{l}{T} = \frac{1}{jr}, \qquad \kappa = \frac{l}{\lambda}\frac{\mathbf{r}}{r} = \frac{1}{jr}\frac{\mathbf{r}}{r}$$

the wave functions (10.9) become

$$\phi_l(\mathbf{x}, t) = \left(\frac{\omega^2}{|\omega|^2}\right)^{k_0 t + \mathbf{k}.\mathbf{x}}$$

The main result of these particular solutions of the wave equations is that the period as well as the wavelength can take only discrete values or, more exactly, integer multiple of some basic length.

11. *Concluding Remarks*

The last section, devoted to physical applications, raises some questions about the viability of the assumptions of Section 6 (remember the strong conditions on the rational periodic functions and the rotation matrices with integer elements). A way of avoiding these restrictions could be to relax assumption (1), by choosing a different field, as was done by Ahmavaara (1965, 1966).

A necessary task that should be undertaken is the application of the lattice model to the area of quantum mechanics and quantum field theory, in a way similar to that proposed by Bopp (1967). In our case the rational character of the wave functions imposes stronger conditions. Nevertheless the Cayley parametrization of the Lorentz group makes possible the invariance of the cubic structure of the lattice without taking the limit, as in Bopp's paper, of an infinite number of points. This correspondence between the continuous and discrete case is also possible in the case of the generators of the Lorentz group, because the matrices S of the Cayley decomposition of semisimple groups satisfy the same commutation relations as the infinitesimal generators, as was shown in Section 5. We can still keep the Lie algebra of the operators in quantum mechanics associated with finite transformations, in the sense of Cayley generators. These operators will be associated with physical observables, but now they will have a discrete spectrum, due to the rational character of their representation.

Finally, some philosophical considerations seem to be unavoidable, although the geometrical and physical assumptions have been stated without them. If the space-time lattice is the fundamental reality of the world, it should be considered the platform where all the events take place. In other words, the world lattice means that there exists an absolute space-time although, from the physical point of view, one observer cannot decide whether his system of reference is at rest and parallel with respect to the world lattice or is moving with respect to it and in an inclined direction with respect to the three basic orthogonal axes.

The assumption of a space-time cubic lattice brings out some other problems which can be discussed, at least, in a philosophical sense. If this lattice is not only a mathematical model but an objective reality, is there any reason by which the fundamental points of the lattice are arranged in this particular cubic structure? If the size of this lattice is finite, as was claimed by Bopp (1967), are the spatial points of two limiting surfaces, which are in opposite sides, connected in such a way that the space can be considered as infinite in any direction? Should not the clear distinction between the world lattice and the particular entities acting on it require the introduction of a new variable responsible for the successive actions produced by the individual entity and expressed through the space-time variables? This idea seems parallel to the introduction of a dynamical variable made by Feynman (1949), and also by Aghassi *et al.* (1970, 1971), and recently by Hurwitz and Piron (1973), who claim that this new variable is necessary for the complete description of the evolution of the physical system.

These questions and other philosophical reflections should be taken seriously and will require a more thorough study.

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